nally more complex than the usual uni-direction adiabatic/isothermal flat plate flows: it is necessary to start with the exact equations distinguishing between Favre and Reynolds decomposed variables and recognizing that their respective Taylor series expansions are related by $v_j = u_j + < v_j >$.

It is typical of current compressible $k-\varepsilon$ and Reynolds stress models to neglect some or all of these contributions to the mean momentum, mean energy and Reynolds stress equations. The retention of the mass flux terms will be important in complex compressible turbulent flows: these include flows in which there are mean density gradients due to large Mach number or combustion, separation or reattachment (inflection points), cold wall boundary conditions, mean dilatation, shocks, adverse pressure gradients, or strong streamwise accelerations.

References

Dinavahi, S.P.G., C. D. Pruett (1993). Analysis of direct numerical simulation data of a Mach 4.5 transitional boundary layer flow. ASME Fluids Engineering Conference, June 21-24, Washington DC.

Dinavahi, S.P.G., J.R. Ristorcelli, G. Erlebacher (1993). Some turbulence statistics relevant to compressible turbulence modeling. To appear in the Proceedings of the ICASE/LaRC Workshop on Transition, Turbulence and Combustion.

Ristorcelli, J.R. (1993). A representation for the turbulent mass flux contribution to Reynolds-stress and two-equation closures of compressible turbulence. ICASE Report No. 93-88, NASA CR 191569, submitted *Phys. Fluids.*

Ristorcelli, J.R., S.P.G. Dinavahi (1993). Testing a model for the turbulent mass flux. To appear in the Proceedings of the ICASE/LaRC Workshop on Transition, Turbulence and Combustion.

Reynolds stress equations can be written in the following form

$$
\begin{aligned}
D/Dt\,(<\rho>\{v_i v_j\}) = \; & -<\rho>\{v_i v_p\}V_{j},_p - <\rho>\{v_j v_p\}V_{i},_p + \prod_{ij} + 2/3 <pd>\delta_{ij} \\
& -[<pv_i>\delta_{pj} + <pv_j>\delta_{ip} + <\rho>\{v_i v_j v_p\}],_p \\
& + 2[<\mu><v_j v_i>,_p - <\mu><v_i d>\delta_{jp}/3 - <\mu><v_j d>\delta_{ip}/3],_p \\
& -2/3\varepsilon_t\delta_{ij} - [<v_j><\sigma_{ip}> + <v_i><\sigma_{jp}>],_p \\
& + <v_j>[-P,_i + \Sigma_{ip},_p + <\sigma_{ip}>,_p] + <v_i>[-P,_j + \Sigma_{jp},_p + <\sigma_{jp}>,_p] \\
& - <\rho>\varepsilon_{ij}^{d} - 2/3 <\rho>(\varepsilon_s + \varepsilon_d)\delta_{ij}
\end{aligned}
$$

where the definition of the stress tensor σ_{ij} has been used to expand $[<v_j\sigma_{ip}> + <v_i\sigma_{jp}>]$ in the viscous transport terms and $d = v_p,_p$. Except for neglecting correlations between the fluctuating Favre velocity and its vorticity the final form of the Reynolds stress equations, given above, is exact.

Conclusions

The mass fluxes or Favre fluctuation means appears in several places in the compressible turbulence equations. They contribute to 1) the viscous terms in the mean momentum equation, and in the mean energy equation they contribute to 2) the viscous terms, 3) the pressure work terms, 4) the viscous work terms, and the 5) pressure flux which is coupled to the mass flux through the equation of state. Modeling $U_i \simeq V_i$ is equivalent to neglecting the mass flux in five different locations in the mean equations. In the Reynolds stress equations the mass flux contributes to 6) the viscous diffusion of the Reynolds stresses, which only recognized when the viscous terms are properly distinguished into their Reynolds and Favre variable components. The mass flux determine the importance of two production mechanisms 7) one due the acceleration of the mean flow and 8) the other due to viscous effects associated with the Favre fluctuation mean and also it contributes to 9) the pressure flux. A general algebraic model for the $<v_i>$, derived from first principles, and suitable for complex flows of engineering interest has been derived in Ristorcelli (1993) and tested in Ristorcelli and Dinavahi (1993).

The viscous terms in the Favre-averaged Reynolds stress equations have been systematically derived, identified and segregated. The source of terms associated with anisotropy and inhomogeneity of the dissipation, essential to consistent near-wall modeling, have been shown. The second essential point is that the dissipation is a function of u_i fluctuations about the Reynolds mean, while the Reynolds stress equation is an evolution equation for statistics of v_i fluctuations around the Favre mean. The near-wall Taylor series expansions of these two quantities, in a general compressible flow, are different. Near-wall asymptotics must recognize these facts when relating Favre type variables $\{v_i v_j\}$, in which the problem is posed, to dissipation type quantities which are carried in Reynolds variables. This point is crucial for any consistent general near-wall model development in flows only nomiterms arising from viscous surface terms appear naturally in u_i variables while the problem is posed in the Favre v_i variables.

The dissipation type terms, which require closure, are kept in u_i variables. The fact that $v_i = u_i + < v_i >$ is used to segregate the terms into ones that require modeling and ones that are carried as dependent variables in the closure scheme. The identities $< u_{i,p}\, u_{p,i} >=< u_i u_p >_{,ip} -2 < u_{i,i}\, u_p >_{,p} + < u_{i,i}\, u_{p,p} >$ and $u_{i,p} = u_{p,i} - \epsilon_{ijp}\omega_j$ where the vorticity is $\omega_i = \epsilon_{ijk} u_{j,k}$ are used to write the trace of the dissipation terms as

$$< u_{j,p}\, \sigma^u_{jp} >=< \mu > [< \omega_k \omega_k > +2(< u_j u_p >_{,jp} -2 < u_{k,k}\, u_p >_{,p}) + 4/3 < u_{k,k}\, u_{p,p} >].$$

Here $< \omega_j \omega_j >=< u_{j,p}\, u_{j,p} > - < u_{j,p}\, u_{p,j} >=< u_{j,p}\, u_{j,p} > - < u_q u_p >_{,qp} +2 < u_{q,q}\, u_p >_{,p} - < u_{q,q}\, u_{p,p} >$. In a homogeneous compressible turbulence $< \omega_j \omega_j >=< u_{j,p}\, u_{j,p} > - < u_{j,j}\, u_{p,p} >$ which reduces to the usual expression in an incompressible homogeneous turbulence. Defining the positive definite scalar dissipation quantities, the solenoidal dissipation $< \rho > \varepsilon_s =< \mu >< \omega_j \omega_j >$ and the dilatational dissipation $< \rho > \varepsilon_d = 4/3 < \mu >< u_{p,p}\, u_{q,q} >$ the trace can then be written as

$$< u_{j,p}\, \sigma^u_{jp} >= < \rho > (\varepsilon_s + \varepsilon_d) \ + \varepsilon_t$$

where $\varepsilon_t = 2 < \mu > [< u_j u_p >_{,j} -2 < u_{k,k}\, u_p >]_{,p}$ is a scalar transport term that comes from the dissipation type terms. The decrease of the Favre kinetic energy, for a homogeneous turbulence is then written as $D/Dt\ (< \rho > \{v_j v_j\}) = -2 < u_{j,p}\, \sigma^u_{jp} >= -2 < \rho > (\varepsilon_s + \varepsilon_d) - 2\varepsilon_t$ where $\varepsilon_t = 0$ Note that ε_t is not defined as per unit mass quantity to emphasize that an equation need not be carried for it: the substitution $u_i = v_i - < v_i >$ shows that it can be written, except for the correlation with the divergence, in terms of the mass flux and the Reynolds stresses for which equations are carried. In general ε_t is either positive or negative: in a homogeneous turbulence ε_t is zero while in the near-wall region it makes a nonnegligible contribution to the energy budget. The point is that in an inhomogeneous turbulence it is necessary to recognize the contributions to $< u_{j,p}\, \sigma^u_{jp} >$ that are functions of gradients of the Reynolds stresses rather than hiding these terms in the dissipation. If the trace is added and subtracted the dissipation-type terms can be rewritten as

$$- < u_{j,p}\, \sigma^u_{ip} > - < u_{i,p}\, \sigma^u_{jp} >= - < \rho > \varepsilon^d_{ij} - 2/3 < \rho > (\varepsilon_s + \varepsilon_d)\delta_{ij} - 2/3 \varepsilon_t \delta_{ij}$$

where the term $< \rho > \varepsilon^d_{ij}$ has zero trace. The viscous terms are now manipulated into their final form. Using $\sigma^u_{ij} = \sigma_{ij} - < \sigma_{ij} >$, which follows from $u_i = v_i - < v_i >$, allows the viscous transport terms to be rewritten as $< v_i \sigma^u_{jp} >=< v_i \sigma_{jp} > - < v_i >< \sigma_{jp} >$. The and where there are gradients in the mean density, the $< v_i >$ are important and need to be carried. Data from $Ma = 4.5$ DNS computations of Dinavahi and Pruett (1993) in unidirectional developing wall bounded flow indicate that the approximation of $U_i \simeq V_i$ in the wall bounded flow is inadequate. This is a nominally simple flow, in comparison to those of practical interest, in which there is a four-fold variation of the mean density over the boundary layer. In data taken from that simulation, shown in Figure 1, it was quite unexpectedly found that $< v_2 >$ is larger than either U_2 and V_2. It is large enough to cause U_2 and V_2 to have different signs. This indicates that the mean fluid particle transfer is in a direction opposite to the net momentum transfer in flows with mean density gradients. Moreover, the contribution of the Favre fluctuation mean to the total viscous stress was found to be as large as one third the contribution of the Favre mean viscous stress, in the near wall portions of the turbulent boundary layer. In flows with separation and re-attachment the second derivative of the mean flow vanishes leaving, in the viscous terms, the second derivative of the Favre fluctuation mean. Clearly this will be an important term when there is a turbulent mass flux due to mean density gradients near the point of separation.

The mean energy equation is subject to similar deficiencies. The exact equation for the mean total energy, internal plus kinetic, is

$$\begin{aligned}(<\rho> E)_{,t} + (<\rho> V_p E)_{,p} = & -[PV_p + P < v_p > + < pv_p >]_{,p} + [< \sigma_{pk} v_k > - < \rho > \{ev_p\}]_{,p} \\ & + [\Sigma_{pk} V_k + < \sigma_{pk} > V_k + \Sigma_{pk} < v_k >]_{,p} - Q_{p,p}\,.\end{aligned}$$

It is not unusual in compressible turbulence models to see several, if not all, of the terms involving the mass flux, $P < v_k >$, $< \sigma_{pk} > V_k$, or $\Sigma_{pk} < v_k >$ dropped because the present models for these terms destabilize computations or because they are, in the spirit of $U_i \simeq V_i$, assumed negligible.

The Reynolds stress equations

The second moment equations for a compressible flow, are written without approximation, and after some manipulation, as

$$\begin{aligned}D/Dt\,(<\rho> \{v_i v_j\}) = & -<\rho> \{v_i v_p\} V_{j,p} - <\rho> \{v_j v_p\} V_{i,p} + \prod_{ij} + 2/3 < p v_{k,k} > \delta_{ij} \\ & -[< pv_i > \delta_{pj} + < pv_j > \delta_{ip} + < \rho > \{v_i v_j v_p\} - < v_j \sigma^u_{ip} > - < v_i \sigma^u_{jp} >]_{,p} \\ & + < v_j > [-P_{,i} + \Sigma_{ik,k} + < \sigma_{ik} >_{,k}] + < v_i > [-P_{,j} + \Sigma_{jk,k} + < \sigma_{jk} >_{,k}] \\ & - < u_{j,p}\, \sigma^u_{ip} > - < u_{i,p}\, \sigma^u_{jp} >\end{aligned}$$

where the mean momentum equations have been used and $\sigma^u_{ij} = < \mu > [u_{i,j} + u_{j,i} - 2/3 u_{q,q}\, \delta_{ij}]$. The form of the equations above reflect the following manipulations: 1) The deviatoric part of the pressure-strain correlation is defined $\prod_{ij} = < p(v_{i,j} + v_{j,i}) > -2/3 < pv_{k,k} > \delta_{ij}$. and 2) the identity $v_i = u_i + < v_i >$ has been used to rewrite the transport terms in v_i variables while keeping the dissipation terms in u_i variables. In the Reynolds stress equations the which follows from the definition of the Favre-average of the Favre fluctuation, $\{v_i\} =< \rho^* v_i >< \rho >^{-1} = 0$. Thus, apart from a scaling by the local mean density, the mean of the fluctuating Favre velocity and the mass flux are equivalent quantities. The primes on the fluctuating density have been dropped.

Mathematically $< v_i >$ represents the difference between unweighted and density- weighted averages of the velocity field and is therefore a measure of the effects of compressibility through variations in density. It plays an important role in parameterizing the anisotropic effects of compressibility associated with the mean dilatation and the mean density gradients. Experimentally it is an important and essential quantity that allows numerical results computed in Favre averaged variables to be related to experimental results computed in Reynolds variables. Additional results as well as an equation for the mass flux have been given in Ristorcelli (1993). The present investigation focuses on the role the Favre fluctuation mean plays in the the mean momentum, mean energy and the Reynolds stress equations and shows that the current practice of neglecting it is unnecessary, inconsistent and inadequate.

The mean flow equations

Substituting in the Favre decomposition into the Navier-Stokes equations and time averaging produces, without approximation,

$$(< \rho > V_i)_{,t} + (< \rho > V_p V_i)_{,p} = -P_{,i} + \Sigma_{ij,j}(U) - (< \rho > \{v_i v_p\})_{,p}$$

where $\Sigma_{ij}(U) =< \mu > [U_{i,j} + U_{j,i} - 2/3 U_{q,q}\delta_{ij}]$. Correlations between the fluctuating viscosity and the fluctuating velocity have been neglected. Note that the viscous terms are given in terms of the time-averaged mean velocity while the problem is solved in terms of the Favre-averaged mean velocity. They are related by $U_i = V_i + < v_i >$. The usual assumption used to close the viscous term is that $U_i \simeq V_i$. Thus even the first-order equation is modeled reducing the accuracy of the method *at the very lowest order* in the very region of the most practical (aerodynamic) interest. Part of the appeal of $k - \varepsilon$ or Reynolds stress turbulence models in the incompressible turbulence is that the equations for the mean flow are carried exactly: this can also be done for the compressible turbulence. Substituting $U_i = V_i + < v_i >$ produces the exact equation for the mean momentum equation:

$$(< \rho > V_i)_{,t} + (< \rho > V_p V_i)_{,p} = -P_{,i} + [\Sigma_{ij} + < \sigma_{ij} >]_{,j} - (< \rho > \{v_i v_p\})_{,p}$$

where $\Sigma_{ij} =< \mu > [V_{i,j} + V_{j,i} - 2/3 V_{q,q}\delta_{ij}]$, and $< \sigma_{ij} >=< \mu > [< v_i >_{,j} + < v_j >_{,i} - 2/3 < v_q >_{,q}\delta_{ij}]$. The mean momentum equation is now carried exactly. Note that the additional terms, $< \sigma_{ij} >$, do not contribute to the mean flow equations if the turbulence is either homogeneous or isotropic. In the near-wall region where the viscous terms are important

Introduction

The stress and work terms in the mean momentum, mean energy and the Reynolds stress equations are usually modeled by assuming that the Favre mean velocity is a suitable approximation to the Reynolds mean velocity. This neglects the contribution of the mean of the fluctuating Favre velocity which, related to the turbulent mass flux, quantifies the difference between the Reynolds and Favre mean velocities of the mean flow. As the stress and work terms do not introduce any new unknown quantities in $k-\varepsilon$ or Reynolds stress models (an eddy viscosity expression for the mass flux is typically carried) and because they can be carried exactly with little additional complexity there is no need or justification for any modeling assumptions. The present article derives the exact equations for the mean flow and the Reynolds stresses for a compressible turbulence keeping the neglected difference between the Reynolds and the Favre mean velocities. In so doing it is hoped to make two crucial points clear: 1) that the retention of the mass flux terms in the several places it appears in the mean momentum, mean energy and Reynolds stress equations is essential to the prediction of any nominally complex engineering flows and 2) that a careful distinction between Reynolds-averaged and Favre-averaged variables must be made to properly pose the near-wall problem and insure that the near-wall asymptotics are carried out consistently.

In this exposition upper case letters will be used to denote mean quantities except in the case of the mean density, $<\rho>$, as ρ has no convenient upper case form. The expectations will be indicated using the angle brackets for time-means, eg. $<v_iv_j>$, and the curly brackets for the density-weighted or Favre-means, eg. $\{v_iv_j\}$, where $\{v_iv_j\} = <\rho^* v_iv_j><\rho>^{-1}$ and the asterisk denotes the full field, $\rho^* = <\rho> + \rho'$. The dependent variables are decomposed according to

$$\begin{aligned} u_i^* &= U_i + u_i && \text{where } <u_i> = 0 \\ u_i^* &= V_i + v_i && \text{where } \{v_i\} = 0 \\ \rho^* &= <\rho> + \rho' && \text{where } <\rho'> = 0 \\ p^* &= P + p && \text{where } <p> = 0. \end{aligned}$$

As both the Reynolds-mean and the Favre-mean velocities appear in the evolution equations for a compressible turbulence it is necessary to carry both the Favre and the Reynolds decompositions of the velocity field. The mean of the fluctuating Favre velocity, $<v_i>$, characterizes the difference between the Favre mean velocity and Reynolds mean velocities, V_i and U_i, as well as the difference between the instantaneous fluctuating velocities:

$$\begin{aligned} U_i &= V_i + <v_i> \\ u_i &= v_i - <v_i>. \end{aligned}$$

The Favre fluctuation mean, a first-order moment, is related to the turbulent mass flux, a second order moment, by

$$<\rho v_i> = \; - <\rho><v_i>$$

CARRYING THE MASS FLUX TERMS EXACTLY IN THE FIRST AND SECOND MOMENT EQUATIONS OF COMPRESSIBLE TURBULENCE

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ABSTRACT

In compressible turbulence models it is assumed that the Favre-mean velocities are suitable approximations to the Reynolds-mean velocities in order to close unknown terms. This neglects, in the mean momentum and energy equations, the contribution to the stress and work terms by the mean of the fluctuating Favre velocity, a quantity proportional to the turbulent mass flux. As the stress and work terms do not introduce any new unknown correlations requiring closure in either $k - \varepsilon$ or Reynolds stress closures and because the exact form of the terms can, with little additional work, be carried there is no need to make any modeling assumptions. In the Reynolds stress equations the viscous terms appear naturally in Reynolds variables while the problem is posed in Favre variables. In the process of splitting the viscous terms into the viscous transport terms, carried in Favre variables, and the dissipation terms, carried in Reynolds variables, important contributions from the mass flux appear. The accurate accounting of these terms is important for any consistent near wall modeling and the retention of the mass flux terms is important in complex compressible turbulent flows.

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